Critical behavior of vorticity in two-dimensional turbulence

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We point out some similarities between the statistics of high Reynolds number turbulence and critical phenomena. An analogy is developed for two-dimensional decaying flows, in particular by studying the scaling properties of the two-point vorticity correlation function within a simple phenomenological framework. The inverse of the Reynolds number is the analog of the small parameter that separates the system from criticality. It is possible to introduce a set of three critical exponents; for the correlation length, the autocorrelation function, and the so-called susceptibility, respectively. The exponents corresponding to the well-known enstrophy cascade theory of Kraichnan and Batchelor are, remarkably, the same as the Gaussian approximation exponents for spin models. The limitations of the analogy, in particular the lack of universal scaling functions, are also discussed. [S1063-651X(99)12312-2]

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I. INTRODUCTION

Turbulence is characterized in a generic way by self-similar spectra, a feature that is also found in other physical situations, such as critical phenomena or random walks. In real space, this property is usually the signature of well-known statistical mechanisms such as the inverse power law decay of two-point correlation functions and the non-Gaussian behavior of the relative dispersion of random variables of the problem. Some attempts have been made to relate scaling in turbulence with critical phenomena and field theories [1]. More recently, experiments have shown that the power consumption in closed turbulent flows and the total magnetization of spin systems at criticality have similar, and to some extent, universal, probability distribution functions [2]. The two-point velocity correlation function has been a quantity extensively studied in hydrodynamic turbulence, mostly under the related form of velocity increments [3]. In practice, a real space experimental and numerical analysis of eventual scaling laws is a difficult, space and time consuming task, because the asymptotic regimes are reached very slowly as the Reynolds number goes to infinity. On the other hand, relatively few studies have been devoted to the characterization of the spatial structure of the vorticity ($\boldsymbol{\omega}=\nabla\times\mathbf{u}$). Although, in Fourier space, the enstrophy spectrum of an incompressible flow is very simply related to the kinetic energy spectrum, the vortex statistics and correlations in real space are not so simply related to their velocity counterparts, but have distinct properties. This difference is illustrated in a spectacular way in two-dimensional turbulence, where the enstrophy $\langle\boldsymbol{\omega}^2(\mathbf{x})\rangle$ cascades toward small scales while the kinetic energy $\langle\mathbf{u}^2(\mathbf{x})\rangle$ follows an inverse cascade to large scales. This is essentially due to the absence of vortex stretching in two dimensions. Because of this property, the enstrophy obeys a conservation equation, and $\langle\boldsymbol{\omega}^2(\mathbf{x})\rangle$ is dissipated even in the very low viscosity limit, contrary to the energy [3,4].

A vortex is one of the simplest kinds of structure generated by hydrodynamic instabilities; an isolated structure is likely to produce slowly decaying flow fields in space (typically with $1/r$, $1/r^2$ dependence) and consequently long range velocity correlations. Physically, a flow can be structured even if the vorticity vanishes nearly everywhere, as in the case of an assembly of point vortices in two dimensions. Hence, the study of vorticity distribution functions should reveal a deeper, less intuitive degree of order in flows. This occurs in two-dimensional turbulence, where the vorticity tends to concentrate in spatially extended structures, of particlelike character. Indeed, numerical studies of two-dimensional decaying turbulence have shown that the long time dynamics is dominated by a few large coherent axisymmetric vortices that contain most of the enstrophy [5]. Dynamically, it is commonly believed that these structures are the result of many vortex mergers. Such flows can often be well described by a modified Hamiltonian, deterministic model of point vortices [6]. These rather well isolated structures of well defined shape and size have the advantage of simplifying the picture of the flow, because the organization of initially numerous incoherent vortices into blobs reduces the number of relevant degrees of freedom.

In this study, following a route similar to the analysis by Babiano *et al.* [7] of velocity structure functions, we characterize the vorticity structure in real space for two-dimensional homogeneous unforced turbulence, keeping in mind its possible representation in terms of assemblies of vortices. We consider the normalized vorticity autocorrelation function, and calculate it assuming a phenomenological self-similar energy spectrum in the inertial range, $E(q)\sim q^{-\mu}$. In experiments or numerical studies, μ is usually found to be around 3 [8], the value predicted by the Kraichnan [9] and Batchelor [10] reference theory, hereafter referred to as KB, or steeper ($\mu\simeq 4$).

In the next section, we show that the vorticity autocorrelation function exhibits distinct behaviors depending on the value of the exponent μ. In Sec. III, we show that it is possible to draw an analogy between two-dimensional turbulence and systems of spins near criticality for values of μ larger than 1. The correlations can indeed be written under the form of a scaling law involving a diverging correlation length when the Reynolds number goes to infinity. The critical exponents that correspond to KB theory precisely coincide with the Gaussian exponents of spin models or random walks. Conclusions including a brief discussion of the three-dimensional case are presented in Sec. IV.

II. VORTICITY POWER SPECTRA

Let us consider the vorticity autocorrelation function

$$R_\omega(\mathbf{r})=\frac{\langle \boldsymbol{\omega}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x}+\mathbf{r})\rangle}{\langle \boldsymbol{\omega}^2(\mathbf{x})\rangle}, \tag{2.1}$$

that has been normalized to one at $\mathbf{r}=\mathbf{0}$. R_ω can be expressed as a function of the Fourier transform of the vorticity

$$\boldsymbol{\omega}(\mathbf{q})=\int d\mathbf{x}\boldsymbol{\omega}(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}, \tag{2.2}$$

through the relation

$$R_\omega(\mathbf{r})=\frac{\int d\mathbf{q}\langle\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(-\mathbf{q})\rangle e^{i\mathbf{q}\cdot\mathbf{r}}}{\int d\mathbf{q}\langle\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(-\mathbf{q})\rangle}. \tag{2.3}$$

The vorticity power spectrum $\langle\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(-\mathbf{q})\rangle$ can easily be related to the kinetic energy spectrum $E(q)$ of the flow $\mathbf{u}$. In two dimensions one gets, in the homogeneous isotropic case

$$E(q)=\frac{q}{4\pi V}\langle\mathbf{u}(\mathbf{q})\cdot\mathbf{u}(-\mathbf{q})\rangle, \tag{2.4}$$

where V is the volume of the flow. If the flow is incompressible, one has the identity $\langle\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(-\mathbf{q})\rangle=q^2\langle\mathbf{u}(\mathbf{q})\cdot\mathbf{u}(-\mathbf{q})\rangle$, so that

$$\langle\boldsymbol{\omega}(\mathbf{q})\cdot\boldsymbol{\omega}(-\mathbf{q})\rangle=4\pi VqE(q). \tag{2.5}$$

Replacing the identity (2.5) in expression (2.3) and integrating over the angular variable gives

$$R_\omega(r)=\frac{\int dq q^2E(q)J_0(qr)}{\int dq q^2E(q)}, \tag{2.6}$$

where J_0 is the Bessel function of the first kind. In the following, one assumes that the energy spectrum is self similar in the inertial range, which extends between an integral scale l at which energy is (initially) injected and an enstrophy dissipation scale a:

$$E(q)\sim q^{-\mu},\quad 2\pi/l<q<2\pi/a. \tag{2.7}$$

The spectrum is supposed to vanish at length scales smaller than a. If the observation scale r is much smaller than l, one can assume in addition that $E(q)=0$ for $q<2\pi/l$. (We discuss in Sec. IV the modifications introduced by considering the part of the spectrum that extends beyond the inertial range.) Note that in the general case, the scale l, and hence the integral Reynolds number, may increase with time, as in freely decaying processes [4]. This is not a restriction to our analysis, which deals with instantaneous spectra. Relation (2.6) simply turns into

$$R_\omega(r)=\left(\int_{q_l}^{q_a}dq q^{-(\mu-2)}J_0(qr)\right)\left(\int_{q_l}^{q_a}dq q^{-(\mu-2)}\right)^{-1}, \tag{2.8}$$

where

$$q_l=2\pi/l,$$
$$q_a=2\pi/a. \tag{2.9}$$

Note that the normalized velocity autocorrelation function $R_u(r)$, has the same expression as Eq. (2.8), replacing μ by $(\mu+2)$. It can be seen that both the numerator and the denominator of Eq. (2.8) diverges when $l\to\infty$ if $\mu\geqslant 3$. Hence, we expect the behavior of $R_\omega(r)$ for $\mu>3$ to differ from its behavior for $\mu<3$, while the analytical form of R_u does not undergo an abrupt change at this point. In the following we discuss the different cases encountered.

A. Case $\mu>3$

To study the correlations in the interval $r\gg a$, we can replace the upper limits of the integrals of Eq. (2.8) by infinity. Thus, $R_\omega(r)$ scales as a function of r/l only, and can be rewritten as

$$R_\omega(r)=1-(\mu-3)q_l^{\mu-3}\int_{q_l}^{\infty}dq q^{-(\mu-2)}[1-J_0(qr)],$$
$$a\ll r. \tag{2.10}$$

If $3<\mu<5$, the integral of Eq. (2.10) is finite as q_l goes to zero. After the variable change $x=qr$, one obtains the first order expansion in r/l,

$$R_\omega\simeq 1-\frac{\Gamma[(5-\mu)/2]}{\Gamma[(\mu-1)/2]}\left(\frac{\pi r}{l}\right)^{\mu-3},\quad a\ll r\ll l; 3<\mu<5, \tag{2.11}$$

where Γ denotes the Gamma function. If $5<\mu<7$, we use the second order expansion of the Bessel function $J_0(x)=1-x^2/4+O(x^4)$, and replace in Eq. (2.10) $[1-J_0(x)]$ by $[1-x^2/4-J_0(x)]+x^2/4$. We obtain

$$R_\omega(r)\simeq 1-\frac{\mu-3}{4(\mu-5)}(q_lr)^2-C_\mu(q_lr)^{\mu-3},\quad 5<\mu<7, \tag{2.12}$$

with

$$C_\mu=(\mu-3)\int_0^\infty dx x^{-(\mu-2)}[1-x^2/4-J_0(x)]. \tag{2.13}$$

The leading behavior of $R_\omega(r)$ is given by the r^2 term. It is easy to show that this property stays valid for all $\mu>5$:

$$R_\omega(r)\simeq 1-\frac{\mu-3}{4(\mu-5)}(q_lr)^2,\quad a\ll r\ll l. \tag{2.14}$$

The results displayed in Eqs. (2.11) and (2.14) are the analogues of those obtained by Babiano *et al.* [7] for the velocity structure functions (in the cases $1<\mu<3$ and $\mu>3$, respectively). The power-law exponent of the corrections for small separation distances does not depend on μ for $\mu>5$. The first order expansions (2.11) and (2.14) show that the characteristic length ruling the decay of R_ω is q_l^{-1}. We conclude that the vorticity correlation length is of order of the integral scale l.

B. Case $3/2<\mu\leqslant 3$

For energy spectra less steep than -3, the denominator of expression (2.8) is well defined as l goes to infinity, but diverges when a goes to zero. In turn, if $3/2<\mu<3$, the numerator of Eq. (2.8) does not necessitate infrared nor ultraviolet cutoffs to be finite, provided that $r>0$. This implies that in the limit where l/a is large, $R_\omega(r)$ can be much smaller than one even if $r\ll l$.

If $a\ll r\ll l$, one can set $q_l=0$ and $q_a=\infty$ in the numerator of Eq. (2.8). Using properties of Bessel functions [11], one obtains

$$R_\omega(r)\simeq\frac{\Gamma[(5-\mu)/2]}{\Gamma[(\mu-1)/2]}\left(\frac{\pi r}{a}\right)^{-(3-\mu)}. \tag{2.15}$$

The correlation function decays algebraically with the distance r. The value $\mu=3$ separates two different scaling regimes of R_ω. At the transition, the numerator of Eq. (2.8) behaves as ln(l/a), and making the variable change $x=qa$, one obtains after a simple calculation:

$$R_\omega(r)=1-\frac{1}{\ln(l/a)}\int_0^{2\pi}dx\,x^{-1}\left[1-J_0\left(x\frac{r}{a}\right)\right],\quad \mu=3. \tag{2.16}$$

When the separation distance is much larger than the dissipation scale, the correlations decay logarithmically

$$R_\omega(r)\simeq\frac{\ln(l/r)-\ln\pi-0.577}{\ln(l/a)},\quad a\ll r\ll l;\mu=3. \tag{2.17}$$

C. Case $\mu\leqslant 3/2$

For values of μ lower than 3/2, both the denominator and the numerator of Eq. (2.8) diverge when a goes to zero. In order to avoid convergence problems due to oscillations when integrating the Bessel function J_0 over a large but finite interval, it is now necessary to replace the discontinuous energy spectrum (2.7) by a continuous spectrum that models better the effects of viscous dissipation [12], for instance,

$$E(q)\sim\begin{cases} q^{-\mu}e^{-q^2/q_a^2} & \text{if } q>q_l,\\ 0 & \text{if } q<q_l.\end{cases} \tag{2.18}$$

In Eq. (2.18), the dissipative spectrum at large wave numbers $q>q_a$ is described by a Gaussian function [13]. Setting $q_l=0$, relation (2.6) yields

$$R_\omega(r)=\frac{2}{\Gamma[(3-\mu)/2]}q_a^{-(3-\mu)}\int_0^\infty dq\,e^{-q^2/q_a^2}q^{2-\mu}J_0(qr), \tag{2.19}$$

that can be recast as

$$R_\omega(r)={}_1F_1\left(\frac{3-\mu}{2},1;-\frac{q_a^2r^2}{4}\right), \tag{2.20}$$

where ${}_1F_1$ is a hypergeometric function. For $q_ar\gg 1$, and if $\mu\neq 1$, R_ω behaves as [11]

$$R_\omega(r)\simeq\frac{1}{\Gamma[(\mu-1)/2]}\left(\frac{\pi r}{a}\right)^{-(3-\mu)}. \tag{2.21}$$

If $\mu>1$, the above expression is positive and does not differ from the behavior displayed by Eq. (2.15). In turn, if $\mu=1$, the prefactor of the algebraic dependence in Eq. (2.21) vanishes and relation (2.20) simply reduces to a short range function

$$R_\omega(r)=e^{-(\pi r/a)^2}. \tag{2.22}$$

The energy spectrum $\mu=1$ is known to correspond to a flow composed by statistically independent point vortices [14]. In that limit, it is easy to show that the vorticity correlations reduce to

$$R_\omega(r)\big|_{\mu=1}=\begin{cases}1 & \text{if } r=0,\\ 0 & \text{if } r>0.\end{cases} \tag{2.23}$$

Replacing a by zero in Eq. (2.22), we recover the distribution given by Eq. (2.23).

If $\mu<1$, vortex correlations for large separation distances r are still given by relation (2.21). However, the prefactor $\Gamma[(\mu-1)/2]^{-1}$ is now *negative*. This case is physically very different from the situation $\mu>1$, where $R_\omega(r)$ is always a positive, slowly decaying function. Indeed, for $\mu<1$, $R_\omega(r)$ decreases for small r and has its first zero for $r\sim a$: this is the signature of a short range function, such as Eq. (2.22). We conclude that R_ω is short range and characterized by a correlation length of order a in the range $\mu\leqslant 1$.

III. DISCUSSION

The results derived in the preceding section show that the two-point vorticity autocorrelation function can be recast under scaling forms, such as

$$R_\omega(r)=\left(\frac{r}{a}\right)^{-\eta}f(r/\xi). \tag{3.1}$$

The above relation defines an exponent η and a correlation length ξ. Pair correlations in systems near a critical point satisfy similar scaling laws, where the scaling function f decays, say, exponentially, and ξ diverges as the temperature approaches the critical temperature. In the present context, the vorticity has various spatial structures depending on the slope of the energy spectrum. The correlation length ξ is small and equals the dissipation length a when $\mu<1$, see Eq. (2.22). The case $\mu>1$ is more interesting because, then, the correlation length is large: $\xi=l\gg a$, see Eqs. (2.11) and (2.14). [In the range $1<\mu<3$, if the separation distance r is no longer small compared with l, it is easy to check that expression (2.15) or (2.21) reduces to the form (3.1) with $\xi=l$.] These two distinct regimes are indeed driven by the behavior of the numerator of Eq. (2.8) with respect to the

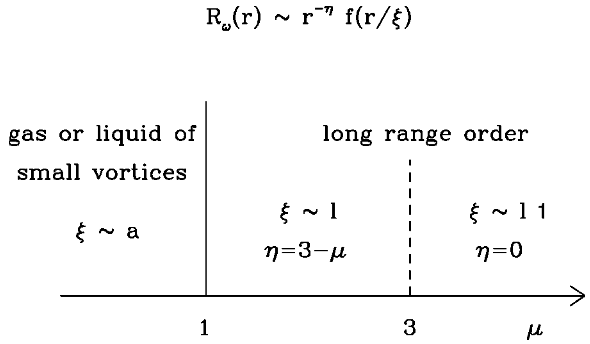

FIG. 1. Scaling forms of the vorticity autocorrelation function in the space of the parameter μ. If the slope of the energy spectrum is lower than 1, R_ω is a short range function. In turn, when $\mu>1$, the correlation length is of order of the integral scale l. Note that for $\mu>3$, the critical exponent η is identically zero.

ultraviolet cutoff. When this cutoff is necessary in order to get a finite integral, the correlations are short range; otherwise, they are long range. This abrupt change is reminiscent of a similar feature encountered in problems of elastic interfaces at thermal equilibrium, namely, the transition from flat to rough surfaces as the surface dimension decreases across the value $d=2$ [15]. In that problem, the rms of the height difference between two well-separated points can be seen as a correlation length. In high dimensions (analogous here to low values of μ), this difference is constant, of order of a small cutoff scale identified with the thickness of the interface (here, the dissipation scale). However, if $d<2$, the height difference is ''macroscopic:'' it depends on the size of the system, i.e., the separation distance between the two points.

The main scaling properties are summarized on the diagram of Fig. 1. Flows with short range correlations can be pictured for instance as a gas, or liquid, of nearly pointlike vortices, with radii of order of the smallest characteristic scale, the dissipation scale a. Using the language of spin models, these structures are also analogous to high temperature states, where a would correspond to the lattice spacing. The part of the diagram where μ is larger than 1 corresponds to vorticity distributions with long range spatial correlations. In the following, we focus on this region and develop an analogy with critical phenomena.

A. Critical exponents

An analogy between a physical problem and critical phenomena requires that some of its statistical properties take a simple scaling form described with a few relevant parameters in a particular asymptotic limit [16]. This can be done, for instance, in the context of random walks, where the analog of the spin-spin correlation is the probability of presence (or the end-to-end distance distribution), and the analogue of the temperature difference from the critical temperature is the inverse of the time (or length of the walk) [17,18].

In turbulence, the integral Reynolds number of a flow is defined as $\mathrm{Re}=vl/\nu_0$, where v is the rms of the velocity and ν_0 the kinematic viscosity. The ratio $(l/a)^d$, where d is the space dimension, is often seen as the number of degrees of freedom of the flow [4]. Phenomenological theories involving dimensional arguments [9,10] usually lead to power law behaviors of l/a with Re, say

$$l\sim a\,\mathrm{Re}^{\nu}. \tag{3.2}$$

Since l is the correlation length in the region $\mu>1$ (see Fig. 1), ν can be understood as a correlation critical exponent. Indeed, as the Reynolds number goes to infinity, the correlation length $\xi=l$ diverges in units of a, and the small parameter that separates the system from criticality is the inverse of the Reynolds number.

The correlation function exponent η of the scaling form (3.1) defines a second critical exponent. $\eta=3-\mu$ in the range $1<\mu<3$, and $\eta=0$ for $\mu>3$. The transition ($\mu=3$) where correlations no longer decay algebraically at infinite Reynolds number coincides with the spectrum of the KB theory. At this particular point, R_ω decays logarithmically as shown by Eq. (2.17). Table I shows the main connections with critical phenomena, for instance, with a system of Ising's spins at temperature slightly above T_c. However, the correspondences are purely statistical: considering few body problems, it is obvious that vortex interaction and dynamics [19] cannot be compared with spin interaction. Concerning this last point, the comparison with spin systems is much more precise in the case of a superfluid [20].

Different theories of turbulence will in general lead to different values of the exponents ν and η. In phenomenological theories, conservation laws and other arguments (such as dimensional arguments) determine the exponent of the energy spectrum, as well as prefactors depending on dissipation rates. Since the size of the inertial range must be consistent with the dissipation rates, once the shape of the spectrum is known and the energy injection scale is fixed, the dissipation scale a [and consequently the value of the exponent ν in relation (3.2)] is unique. Hence, the critical exponents η and ν are generally not independent.

TABLE I. Two-dimensional turbulence and critical phenomena compared.

2D turbulence	Critical phenomena
$R_\omega(r)=\langle\boldsymbol{\omega}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x}+\mathbf{r})\rangle/\langle\boldsymbol{\omega}^2(\mathbf{x})\rangle$	Spin correlations $\langle S(\mathbf{x})S(\mathbf{x}+\mathbf{r})\rangle$
$1/\mathrm{Re}(\to 0)$	$t=\lvert T-T_c\rvert/T_c\ (\to 0)$
$R_\omega(r)\sim(r/a)^{-\eta}f(r/l)$	$\langle S(\mathbf{x})S(\mathbf{x}+\mathbf{r})\rangle\sim r^{2-d-\eta}f(r/\xi)$
Dissipation scale a	Lattice spacing a
Integral length: $l\sim a(1/\mathrm{Re})^{-\nu}$	Correlation length $\xi\sim t^{-\nu}$
$\chi\equiv A^{-1}\langle(\int_A d\mathbf{x}\boldsymbol{\omega}(\mathbf{x}))^2\rangle/\langle\boldsymbol{\omega}^2\rangle\sim a^2(1/\mathrm{Re})^{-\gamma}$	Susceptibility: $\chi\propto\langle(\Sigma_i\mathbf{S}_i)^2\rangle\sim t^{-\gamma}$
Exponents relation: $\gamma=\nu(2-\eta)$	$\gamma=\nu(2-\eta)$
Kraichnan and Batchelor theories: $\eta=0,\ \nu=1/2$	Gaussian result: $\eta=0,\ \nu=1/2$

The KB approach to two-dimensional turbulence assumes that the enstrophy flux through wave number q is independent of q at large q; a dimensional argument leads to $\mu=3$ and $(l/a)^2\sim\mathrm{Re}$ [4]. The critical exponents associated to the KB theory are hence $\eta=0$ and $\nu=1/2$. These exponents are, remarkably, the same as the exponents of the Gaussian approximation in spin models, or the Brownian motion exponents (see Ref. [18]). Although the meaning of this coincidence remains unclear, first, it can be noted that a mean field approximation in the present fluid dynamics context corresponds to the fact of neglecting the fluctuations of the enstrophy dissipation rate during the enstrophy cascade. Secondly, the KB theory is based on two conservation equations: for the kinetic energy $\langle \mathbf{u}^2\rangle$ and for the enstrophy $\langle\omega^2\rangle$. Such a description is incomplete: indeed, in two dimensions the vorticity follows the fluid motion in the inviscid limit, hence any continuous functional $\langle F(\omega)\rangle$ is invariant with time, or must satisfy a conservation equation in the slightly dissipative case. Assuming the Gaussian approximation, $\langle\omega^{2n+1}\rangle=0$ and $\langle\omega^{2n}\rangle=\langle\omega^2\rangle^n$ for any integer n, the conservation of $\langle\omega^2\rangle$ implies the conservation of any functional $F(\omega)=\Sigma_n a_n\omega^n$. Hence, the KB theory ressembles a Gaussian approximation, by focusing only on the second moment of the vorticity.

B. Lack of universality

Although microscopic details (here the dissipation processes at $q>q_a$) disappear in the behavior of $R_\omega(r)$ for large separation distances, the analogy with critical phenomena presented in this section encounters a limitation when looking more closely at the scaling function f in Eq. (3.1). In the case of turbulence, f does not rigorously converge in distinct ''universality classes,'' since it does not depend on the self similar energy spectrum of the inertial range only. The scaling function f essentially depends on the shape of the energy spectrum at low wave numbers. In general, the integral scale is not the largest scale in turbulent flows and coherent structures may emerge from external constraints: the wave numbers lower than $2\pi/l$ contain a non-negligible part of the energy and this part of the spectrum is driven by ''nonuniversal'' conditions such as interactions between large eddies, depending on the system size and boundary conditions. Hence, only short distance scaling is generic in turbulence, as already noted by Eyink *et al.* [1].

Let us consider the modified spectrum

$$E(q)\sim\begin{cases} q_l^{-(\mu_0+\mu)}q^{\mu_0}, & 0<q<q_l,\\ q^{-\mu}, & q_l<q<q_a.\end{cases} \tag{3.3}$$

In two dimensions, the exponent $\mu_0=1$ corresponds to an equipartition kinetic energy spectrum at low wave numbers; the value $\mu_0=3$ has also been proposed for decaying turbulence [21]. It can be easily shown that in the range $r\ll l$, Eqs. (2.15) and (2.20) remain unchanged with the spectrum given by Eq. (3.3). However, if one wishes to compute the scaling function, in the case $\mu>3$ for instance, Eq. (2.10) turns into

$$R_\omega(r)=f\left(\frac{r}{l}\right)=1-\left(\frac{1}{\mu_0+3}+\frac{1}{\mu-3}\right)^{-1}q_l^{\mu-3}\times\int_{q_l}^{\infty}dq\,q^{-(\mu-2)}[1-J_0(qr)], \tag{3.4}$$

$$r\ll l;\mu>3.$$

Hence, the differences introduced by the modified spectrum in the scaling function $f(u)$, defined by Eq. (3.1), already appear at the first order expansion in u: the prefactor is also a function of μ_0. As u grows, the decay of $f(u)$ depends more strongly on the ''nonuniversal'' part.

However, with $E(q)$ given by Eq. (3.3), the leading behavior of $f(u)$ for $u\gg1$ is generically exponential. A famous example is provided by the Ornstein-Zernike (OZ) approximation for critical phenomena, where the spin-spin correlation function reads $\langle S(0)S(r)\rangle\propto\int d^2\mathbf{k}\exp(i\mathbf{k}.\mathbf{r})/(k^2+\xi^{-2})\sim r^{-1/2}\exp(-r/\xi)$ for $r\gg\xi$ [22], corresponding in the present description to $\mu_0=-1$ and $\mu=3$. Recent experimental results on two-dimensional decaying turbulence in soap films have shown clear evidence of spectra with both power-law and inverse power-law regimes, with respective slopes close to the values $\mu_0=1.5$ and $\mu=3$ [8]. Analytical calculations can be performed in the equipartition case $\mu_0=1$ and $\mu=3$: let us rewrite the energy spectrum under the form $E(q)\sim q/(q^4+\xi^{-4})$ with $\xi=l$. The vorticity correlations read

$$R_\omega(r)\propto\int_0^\infty dq\,q^3\frac{J_0(qr)}{q^4+\xi^{-4}}. \tag{3.5}$$

Differing from the OZ result, the asymptotic limit of Eq. (3.5) exhibits oscillations [23]:

$$f(u)\propto\frac{1}{u^{1/2}}\exp(-u/\sqrt{2})\cos(u/\sqrt{2}+\pi/8),\quad u\gg1. \tag{3.6}$$

We now outline a physical interpretation of the scaling function f. As mentioned before, some two-dimensional decaying turbulent flows with a spectrum given by Eq. (3.3) are well pictured by an assembly of well separated vortex blobs with particlelike character. This has been, for instance, very clearly observed in some recent numerical simulations on homogeneous Rossby wave turbulence [24]. Let us assume that these vortices are all of size ξ and with the same circulation modulus. The vorticity distribution inside one blob is thus described by the short distance behavior of the vorticity correlation function, see, e.g., Eq. (2.17). In turn, the long distance behavior of R_ω accounts for the weak correlations between different vortices. Following a route similar to the one presented in Ref. [25], it is easy to show from the definition (2.1) that, if there are the same number of blobs with positive and negative circulation so that the flow has no net overall rotation

$$R_\omega(r)\propto\frac{1}{2}[g_l(r)-g_u(r)],\quad r>\xi. \tag{3.7}$$

Using the notations employed for ionic liquids, g_l represents the pair correlation function between two vortices of same circulation (''like''), and g_u the pair correlation function between two vortices with opposite circulation (''unlike''). The oscillating decaying shape of particle pair correlation functions is a generic feature of liquids, in particular, those that are ionic [26]. The oscillations of R_ω obtained in a particular case through Eq. (3.6), although having a very different physical origin, qualitatively support the idea that the structure of homogeneous assemblies of coherent vortices can be liquidlike.

C. Susceptibility

Since the autocorrelation function R_ω is an averaged quantity, it cannot of course provide by itself all the details on the complexity of vortex structures observed in turbulence, and contained in higher moments. Yet, large coherent vortices can be seen as a particular manifestation of long range order up to the integral scale, similarly to magnetization domains in systems of spins near a critical point. These vortices produce large fluctuations of vorticity on scales smaller than the integral scale, and possibly spatial intermittency. The fluctuations of the total vorticity contained inside an observation window of area A are non-Gaussian (for $\mu>1$) when A is lower than the square of the integral scale. To see this, let us define the circulation

$$\Omega_A=\frac{1}{\langle\boldsymbol{\omega}^2\rangle^{1/2}}\int_A d\mathbf{x}\cdot\boldsymbol{\omega}(\mathbf{x}). \tag{3.8}$$

The standard deviation $\sqrt{\langle\Omega_A^2\rangle}$ of Ω_A from its mean value ($\langle\Omega_A\rangle=0$) can be estimated by considering Ω_A as a sum of random correlated variables. According to the definitions (2.1) and (3.8), one gets

$$\sqrt{\langle\Omega_A^2\rangle}\sim\left(A\int_A d\mathbf{r}R_\omega(\mathbf{r})\right)^{1/2}. \tag{3.9}$$

When R_ω is short range, one obtains the Gaussian behavior

$$\sqrt{\langle\Omega_A^2\rangle}\sim A^{1/2}\quad \mu\leqslant 1, \tag{3.10}$$

at any scale larger than the dissipation scale. In turn, when the correlation length equals the integral scale l, the scaling exponent is non-Gaussian (i.e., larger than 1/2) at scales smaller than l. With the help of Eqs. (3.1) and (3.9), we indeed find

$$\sqrt{\langle\Omega_A^2\rangle}\sim A^{\alpha}\quad (A<l^2) \tag{3.11}$$

with

$$\alpha=1-\eta/4=\begin{cases}(\mu+1)/4, & 1<\mu<3,\\ 1, & \mu>3.\end{cases} \tag{3.12}$$

To derive a critical relation analogous to the one for the susceptibility, we invoke the arguments of Bouchaud and Georges [18] in their statistical interpretation of the general relations between critical exponents. If the area A of the observation window exceeds l^2, one can replace the integration domain of Eq. (3.9) by l^2; multiplying the factor A by l^2/l^2, one gets

$$\sqrt{\langle\Omega_A^2\rangle}=(A/l^2)^{1/2}\langle\Omega_{l^2}^2\rangle^{1/2}\sim(A/l^2)^{1/2}(l^2)^{\alpha},\quad A\gg l^2. \tag{3.13}$$

Hence, the system can be pictured as A/l^2 independent regions of size l at criticality. We define the susceptibility per unit area as

$$\chi=\frac{\langle\Omega_A^2\rangle}{A}. \tag{3.14}$$

When A is larger than the integral scale squared, χ becomes independent of A. Taking into account the dissipation scale a in the calculations, we find from expression (3.13) combined with Eqs. (3.2) and (3.12)

$$\chi\sim a^2(1/\mathrm{Re})^{-\gamma}, \tag{3.15}$$

with the exponent γ satisfying the same relation as in critical phenomena:

$$\gamma=\nu(2-\eta). \tag{3.16}$$

Relation (3.15) shows that the susceptibility diverges when compared with the ''microscopic'' susceptibility a^2. Note that contrary to the lattice spacing in spin models, the dissipation length a may not remain constant while other parameters of the flow are varied. In any case, Re,$\langle\omega^2\rangle$, and a are independent variables.

IV. CONCLUSION

We have studied the two-point vorticity correlation function in two-dimensional turbulent flows characterized by a self-similar energy spectrum within an inertial range limited by two very different length scales. If the slope is steeper than 1, the vorticity distribution presents some similitudes with the magnetization in models of spins near critical point.

On the grounds of phenomenological theories of turbulence, the analogy with critical phenomena is made possible because the ratio between the integral scale (the macroscopic scale at which energy is injected) and the dissipation scale diverges as a power law of the Reynolds number in the large Reynolds number limit. The divergence of susceptibility is associated here with the increasing fluctuations of the circulation taken around a large contour, as the Reynolds number increases. As a surprising result, the (nonindependent) critical exponents corresponding to the Kraichnan [9] and Batchelor [10] (KB) theory are the same as those of the Gaussian approximation result in spin models. A possible connection between these two phenomena could be that the KB theory only considers the first nonvanishing moment of the vorticity (i.e., $\langle\omega^2\rangle$) as an inviscid invariant of the motion, among all the infinite number of possible invariants associated with the vorticity. If one invokes the Gaussian approximation $\langle\omega^{2n}\rangle=\langle\omega^2\rangle^n$ for any integer n, the conservation of $\langle\omega^2\rangle$ is indeed sufficient for any C^∞ functional $F(\omega)$ to be conserved. Another particularity of the KB theory is that the vorticity correlations decay logarithmically for distances larger than the dissipation scale and shorter than the integral scale. The slope of the KB energy spectrum (-3) indeed corresponds to a transition value, where the correlation function no longer decays algebraically at an infinite Reynolds number.

Flows with the same inertial range spectrum may have distinct scaling functions, depending on their large-scale configuration; the nonuniversality of correlations can already be noticed by looking at the prefactor of a first order expansion in r/ξ. However, these functions commonly decay exponentially at distances much larger than the correlation length. On the basis of experimental results [8], we argue that pair correlations between large-scale coherent vortices may be liquidlike.

Most of the statistical remarks made above are not qualitatively modified in three dimensions. It can be shown that for a Kolmogorov $-5/3$ law, the vorticity correlations decay algebraically and satisfy relation (3.1) with $\eta=4/3$ [27]. In the general case, at finite Reynolds numbers, the circulation modulus introduced in relation (3.8) is a sum of random variables if the integrated area extends over many integral scales, see Eq. (3.13). However, it scales anomalously in the inertial range, see Eq. (3.11). This is probably a feature satisfied by other global quantities in turbulent flows, such as the power consumption studied experimentally in Refs. [2,28]. The present analysis is consistent with some of the results presented in these two references: in a closed flow confined to a size of order of one energy injection scale, the power consumption fluctuates strongly. In turn, in an open flow, many large eddies can develop and the fluctuations are Gaussian, according to the central limit theorem.

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